10 persistent homology lecture

Tuesday, March 17, 2020 11:33 AM

Definition 3.1 An affine combination of $\{u_i\}_{i=0}^n$ is a point $x = \sum_{j=0}^n \lambda_i u_i$ s.t. $\sum_{i=0}^n \lambda_i = 1$. An convex combination of $\{u_i\}_{i=0}^n$ is a point $x = \sum_{j=0}^n \lambda_i u_i$ s.t. $\sum_{i=0}^n \lambda_i = 1$ and $\lambda_i \geq 0 \ \forall i \in [n]$

Definition 3.2 Affine and convex hulls

$$\mathrm{aff}(u_0, \dots, u_n) = \left\{ x = \sum_{i=0}^n \lambda_i u_i \;\middle|\; \sum_{i=0}^n \lambda_i = 1 \right\}$$

$$\mathrm{conv}(u_0, \dots, u_n) = \left\{ x = \sum_{i=0}^n \lambda_i u_i \;\middle|\; \sum_{i=0}^n \lambda_i = 1,\ \lambda_i \geq 0 \ \forall i \right\}$$

Ex.

Definition 3.3 $u_0, \dots, u_n$ are **affinely ind.** iff the n vectors $u_i - u_0$ for $1 \leq i \leq n$, are linearly ind.

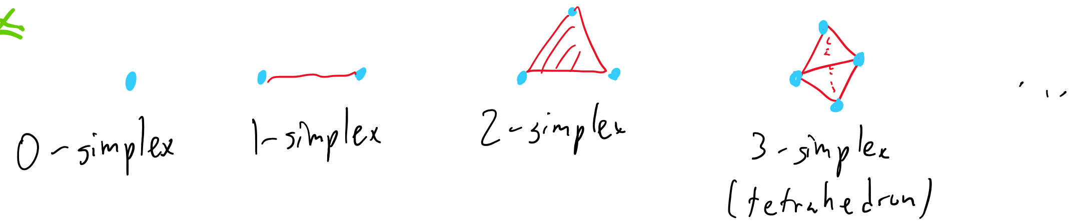
Ex. In $\mathbb{R}^d$, at most $d+1$ affinely ind pts.

Definition 3.4 A **k-simplex** is the convex hull of $k+1$ affinely ind. points

$$\sigma = \mathrm{conv}(u_0, \dots, u_n), \quad \dim(\sigma) = n.$$

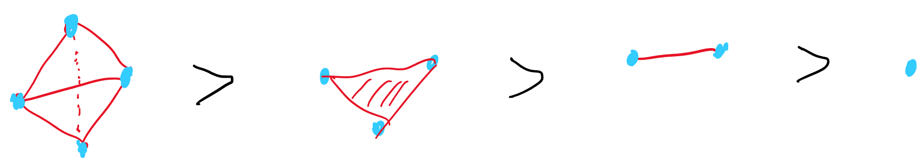
Ex

Ex



Definition 3.5 Given $\sigma = \mathrm{conv}(u_0, \dots, u_n)$, a **face** τ of σ, denoted $\tau \leq \sigma$ is $\tau = \mathrm{conv}(u_{i_1}, \dots, u_{i_m})$, where $\{u_{i_1}, \dots, u_{i_m}\} \subset \{u_0, \dots, u_n\}$. We say that τ is a **proper face** if $m < n$.

Ex.



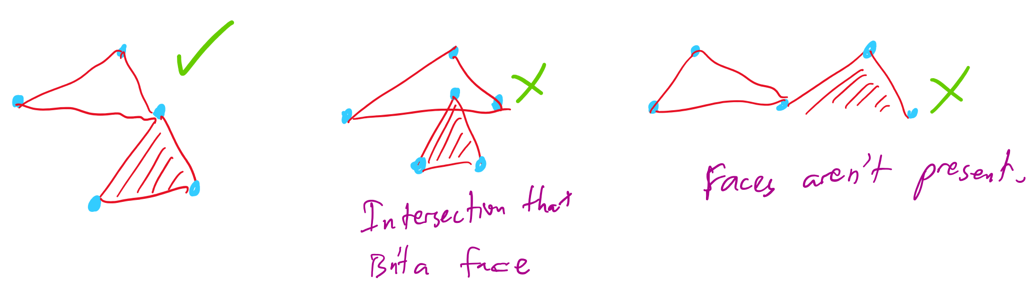
Definition 3.6 A simplicial complex is a finite collection of simplices K s.t.

(1) $\sigma \in K$ and $\tau \leq \sigma \Rightarrow \tau \in K$

(2) $\sigma_1, \sigma_2 \in K \Rightarrow$ either (ii) $\sigma_1 \cap \sigma_2 = \emptyset$ or

(i) $\sigma_1 \cap \sigma_2 \leq \sigma_1$ and $\sigma_1 \cap \sigma_2 \leq \sigma_2$.

Ex.

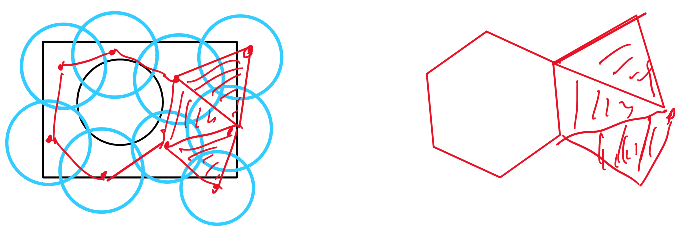


Definition 3.7 An **abstract simplicial complex** is a finite collection of sets A s.t. $\alpha \in A$ and $\beta \subset \alpha$ implies $\beta \in A$.

Definition 3.8 Let X be a topological space. A **cover** of X is a collection of sets $\mathcal{U} = \{U_i\}_{i \in I}$ s.t. $X \subset \bigcup_{i \in I} U_i$.

Definition 3.9 Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of X. The **nerve** of $\mathcal{U}$, $\mathcal{N}(\mathcal{U})$, is the abstract simplicial complex with vertex set I, where a family $\{i_0, \dots, i_k\}$ spans a k-simplex iff $U_{i_0} \cap \dots \cap U_{i_k} \neq \emptyset$.

Ex



Theorem 3.1 (Nerve Thm) Let $\mathcal{U}$ be a finite collection of closed, convex sets in Euclidean space. Then $\mathcal{N}(\mathcal{U})$ and the union of sets in $\mathcal{U}$ have the same homotopy type.

Recall: Given continuous maps $f, g : X \to Y$, a homotopy between f and g is another continuous map $H : X \times [0,1] \to Y$ s.t. $f(x) = H(x, 0)$
$g(x) = H(x, 1)$ $\forall x \in X$.

If $\exists$ such a map H, then $f \simeq g$ (f and g are homotopic).

Two topological spaces X and Y are homotopy equivalent if there are continuous maps $f : X \to Y$ and $g : Y \to X$ s.t.

$$g \circ f \simeq id_X \quad \text{and} \quad f \circ g \simeq id_Y .$$

Definition 3.10 (Cech complex)

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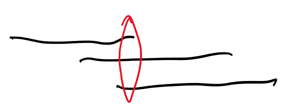
Let X be a finite set of points in $\mathbb{R}^d$. For each $x \in X$, let $B_r(x) = \{y \in \mathbb{R}^d \mid d(x,y) \leq r\}$ be the closed ball centered at x with radius $r \geq 0$. The **Cech complex** of X and r is the nerve of $\{B_r(x)\}_{x \in X}$ i.e.

$$\mathrm{Cech}(X, r) = \{\sigma \subset X \mid \bigcap_{x \in \sigma} B_r(x) \neq \emptyset\}$$

Computing the Cech complex:

Helly's Thm: Let F be a finite collection of closed, convex sets in $\mathbb{R}^d$. Every $d+1$ of the sets have a non-empty intersection iff they all have a non-empty intersection.

Ex.



Proof. Induction over d and number of sets $n = |F|$.

Backward case: If all have non-empty intersection, then clearly any $d+1$ of them do.

Forward case:

Base case: $n = d+1$. Trivial by definition.

Base case: $d = 1$, $\forall n$.

Convex sets on the real line are closed intervals $I_1, \ldots, I_n$.

Start with every pair of intervals intersect.

Let $I_i = [a_i, b_i]$. Then $\bigcap_i I_i = [\max_i a_i, \min_i b_i]$.

If $\max_i a_i > \min_i b_i$, then $\exists\, a_i > b_j$ for some $i \neq j$.

But then $I_i \cap I_j = \emptyset$, so we have a contradiction.

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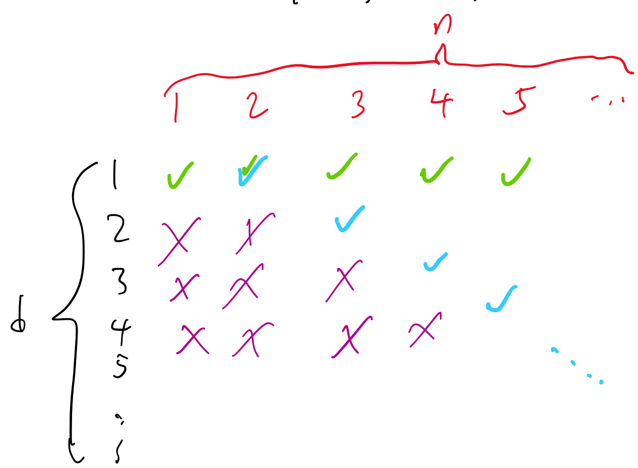


General case: Suppose $\exists\, n > d+1$ closed, convex sets in $\mathbb{R}^d$, denoted $X_1, \ldots, X_n$, form a minimal counterexample, where every $d+1$ of the sets has a common intersection, but not all n sets.

(inductive hypothesis: True $(d, n-1)$, and $(d-1, n)$)

By mimality and the inductive hypothesis,

$$Y_n = \bigcap_{i=1}^{n-1} X_i \text{ is non-empty and disjoint from } X_n.$$

Because Y_n and X_n are closed and convex, $\exists\,(d-1)$-dim. hyperplane h that separates them and is disjoint from both sets.

Let F' be the collection of sets $Z_i = X_i \cap h$, for $1 \le i \le n-1$.

Note: each Z_i is a non-empty $(d-1)$-dim closed, convex set.

By assumption, d of the first $n-1$ sets X_i have a common intersection with X_n

Thus, that common intersection of d sets contains points on both sides of h, since they intersect both Y_i and X_i

$\Rightarrow$ any d sets of $\{Z_i\}$ have a common intersection

$\Rightarrow \bigcap F' \neq \emptyset$ (by inductive hypothesis)

But, $\bigcap F' = \bigcap_{i=1}^{n-1} (X_i \cap h) = Y_n \cap h.$

Contradiction, because Y_i is disjoint from h, proving the claim $\boxed{/\!/}$

Note that a set of balls of equal radius has a non-empty intersection iff their centers lie in a ball of the same radius.

$\Rightarrow$ y belongs to al balls iff $d(x,y) \leq r$ for all centers $x \in X$

Corollary: (Jung's Thm): Let $X \subseteq \mathbb{R}^d$ a finite set. Every $d+1$ points in X are contained in a common ball of radius r iff all points in X are $\boxed{/\!/}$

Let $\sigma \subseteq X$. Then $\sigma \in \mathrm{Cech}(X,r)$ if $\sigma \subseteq B_r(y)$ for some $y \in \mathbb{R}^d$.

Let miniball(σ) be the smallest closed ball containing σ.

The radius of miniball(σ) $< r \iff \sigma \in \mathrm{Cech}(X,r)$.

Note that the miniball is determined by its boundary points, so we can recursively remove non-boundary points.

Algorithm returns the miniball with τ in the interior and ν on the boundary,

def MiniBall(τ, ν):

if $\tau = \emptyset$, then compute the miniball B of ν directly.

else, choose a random $u \in \tau$,

B = miniball$(\tau - \{u\}, \nu)$ remove u from interior

if $u \notin B$, then B = miniball$(\tau - \{u\}, \nu \cup \{u\})$ put u in boundary if necessary.

return B.

Then miniball$(\sigma, \emptyset)$ = miniball(σ)

Let $t_j(n)$ be the expected computational complexity with n points in τ and $j = d+1-|\nu|$ possibly open positions on the boundary

$t_j(0) = 0$

If $n>0$, the the $\mathbb{P}rob(u \notin B) = \mathbb{P}rob($u needs to be a boundary element$)$
$\leq \frac{j}{n}$.

Thus, $t_j(n) \leq \underbrace{t_j(n-1)}_{\text{miniball}(\tau - \{u\}, \nu)} + \underbrace{1}_{u \notin B} + \underbrace{\frac{j}{n} t_{j-1}(n-1)}_{\text{miniball}(\tau - \{u\}, \nu \cup \{u\})}$

$t_0(n) \leq t_0(n-1) + 1 \quad \Rightarrow \quad t_0(n) \leq n$

$t_1(n) \leq t_1(n-1) + 1 + \frac{1}{n} t_0(n-1) \leq t_1(n-1) + 2 \leq 2n$

$t_2(n) \leq t_2(n-1) + 1 + \frac{2}{n} \cdot 2n \leq t_2(n-1) + 5 \leq 5n$

$t_3(n) \leq t_3(n-1) + 1 + 3 \cdot 5 = 16n$

$t_4(n) \leq (4 \cdot 16 + 1) n \leq 5 \cdot 16 \cdot n$

$t_5(n) \leq (6 \cdot 5 \cdot 4 \cdot 4) n \leq 6!\, n$

$\vdots$

$t_j(n) \leq (j+1)!\, n.$

But, $j \leq d+1$, because the ball is entirely determined by at most $d+1$ boundary points, so for constant dimension, the algorithm takes $O(n)$ time to compute a miniball.

Definition 3.11 Let $X \subseteq \mathbb{R}^d$ be a finite set of pts.

The **Vietoris Rips complex** of X and r is defined to be

$VR(X, r) = \{ \sigma \subset X \mid B_r(x_i) \cap B_r(x_j) \neq \emptyset \ \forall x_i, x_j \in \sigma \}.$

The **Vietoris Rips complex** of X and r is defined to be

$$VR(X,r) = \{ \sigma \subset X \mid B_r(x_i) \cap B_r(x_j) \neq \emptyset \ \ \forall x_i, x_j \in \sigma \}.$$

i.e. $VR(X,r)$ contains all subsets of X with diameter no greater than $2r$.

Easy to see $\quad Cech(X,r) \subseteq VR(X,r)$.

In HW, prove $\quad VR(x,r) \subseteq Cech(X, \sqrt{2} \cdot r)$.

Definition: Let K be a simplicial complex. An **i-chain** is a formal sum of i-simplices $\sum c_i \sigma_i$ where $c_i \in \mathbb{F}$ and the sum is taken over all possible i-simplices $\sigma_i \in K$. The set of all i-chains is denoted $C_i(K)$.

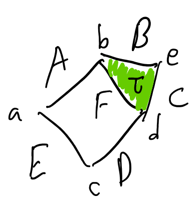
Often, we let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

$C_i(K)$ is a vector space over $\mathbb{F}$, called the vector space of i-chains in K.

Note, the i-simplices form a basis of $C_i(K)$, so

$$\dim(C_i(K)) = \#\ i\text{-simplices}.$$

Ex.



0-simplices $\{a,b,c,d,e\}$

1-simplices $\{A,B,C,D,E,F\}$

2-simplices $\{\tau\}$

$C_0(K) = \langle a,b,c,d,e \rangle \ni a+c+e$

$C_1(K) = \langle A,B,C,D,E,F \rangle \ni A+B+D+E$ $\quad \Big\}$ If $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

$C_2(K) = \langle \tau \rangle \ni \tau$

Definition: (Boundary) Let $\sigma = [u_0, \ldots, u_k]$ be a k-simplex.

The boundary of σ is a map $\partial_k : C_k(X) \to C_{k-1}(X)$

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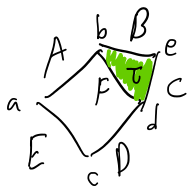
The boundary of σ is a map $\partial_k : C_k(X) \to C_{k-1}(X)$

$$\partial_k \sigma = \sum_{i=0}^{k} [u_0, u_1, \dots, \hat{u}_i, \dots, u_k],$$

where we use the notation $\hat{u}_i$ to indicate that u_i is omitted.

Ex.



$$\partial(\tau) = B + F + C$$

$$\partial(A) = a + b$$

Definition 4.3 (Chain complex)

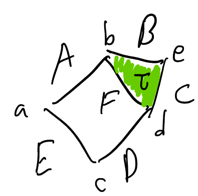
A chain complex is a sequence of chain groups connected by boundary maps.

$$\cdots \xrightarrow{\partial_{i+2}} C_{i+1}(K) \xrightarrow{\partial_{i+1}} C_i(K) \xrightarrow{\partial_i} C_{i-1}(K) \xrightarrow{\partial_{i-1}} \cdots$$

Ex. $\langle \tau \rangle \xrightarrow{\partial_2} \langle A, B, C, D, E, F \rangle \xrightarrow{\partial_1} \langle a, b, c, d, e \rangle \xrightarrow{\partial_0} \emptyset.$

Definition 4.4 (i-cycle)

An i-cycle is an i-chain s.t. $\partial_i c = 0$.



Ex.

$$\partial(C + B + F) = (e + d) + (b + e) + (b + d)$$
$$= 2e + 2d + 2b = 0$$
$$\Rightarrow (C + B + F) \text{ is a 1-cycle.}$$

Definition 4.5 (i-boundary)

An i-chain is an i-boundary if $\exists$ $i+1$-chain $d \in C_{i+1}(K)$ s.t. $c = \partial_{i+1}(d)$.

Ex. $B + C + F = \partial(\tau)$.

Lemma 4.1 (Fundamental lemma of homology)

$\partial_p \circ \partial_{p+1}(d) = 0 \quad \forall p \in \mathbb{Z}$ and for all $p+1$-chains d.

proof. We only need to show this for $(p+1)$-simplex τ. i.e. $\partial_p \circ \partial_{p+1}(\tau) = 0$.

The boundary of $\partial_{p+1}\tau$ consists of all p-faces of τ.

Every $(p-1)$-face of τ belongs to exactly two p-faces,

so $\quad \partial_p(\partial_{p+1}\tau) = 0$.

$\square$